Gauss Elimination and LU Decomposition Example

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System of equations	Associated matrices
$2x_1 - 4x_2 + 2x_3 = 6 \quad (1)$ $4x_1 - 9x_2 + 7x_3 = 20 \quad (2)$ $2x_1 + x_2 + 3x_3 = 14 \quad (3)$	$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 4 & -9 & 7 \\ 2 & 1 & 3 \end{bmatrix}}_{A}$
(Step 1) $(2') = (2)$ - $(1)\times 2$; $(1')=(1)$; $(3')=(3)$ $2x_1 - 4x_2 + 2x_3 = 6 \quad (1')$ $-x_2 + 3x_3 = 8 \quad (2')$ $2x_1 + x_2 + 3x_3 = 14 \quad (3')$	$(2) = 2\times(1) + 1\times(2')$; $(1)=(1')$; $(3)=(3')$ $\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 4 & -9 & 7 \\ 2 & 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}}_{A'}$
(Step 2) $(3'') = (3')$ - $(1')\times 1$; $(1'')=(1')$; $(2'')=(2')$ $2x_1 - 4x_2 + 2x_3 = 6 \quad (1'')$ $-x_2 + 3x_3 = 8 \quad (2'')$ $5x_2 + x_3 = 8 \quad (3'')$	$(3') = 1\times(1') + 1\times(3'')$; $(1')=(1'')$; $(2')=(2'')$ $\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}}_{A'} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''}$

(Step 3)	
(3‴) = (3″) - (2″)× (-5); (1‴)=(1″); (2‴)=(2″)	(3″) = -5×(2″) + ×(3‴); (1″)=(1‴); (2″)=(2‴)
$2x_1 - 4x_2 + 2x_3 = 6 \quad (1''')$ $-x_2 + 3x_3 = 8 \quad (2''')$ $16x_3 = 48 \quad (3''')$	$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix}}_{A'''}$

Thus

$$\begin{aligned} A &= L_1 A' \\ &= L_1 L_2 A'' \\ &= L_1 L_2 L_3 A''' \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix} \quad = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix}}_{U}. \end{aligned}$$